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How Einstein's Theory of Relativity Gives us $E = m\,c^2$ and the Atomic Bomb

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Abstract. The LIGO *direct-detection* of gravitational waves arriving from cosmic sources—now, happily including (as of this meeting!) the merging of two neutron stars—opens a new chapter in our understanding of physics itself: for General Relativity, conceptually so extremely simple, has robustly produced predictions that have invariably been found to be correct when tested. My poster (page 3 of this paper) is intended for high school students who have just learned simple algebra. My derivation of the famous $E = m\,c^2$ from the Pythagorean Theorem necessarily requires an algebraic expansion that is due to Newton, but apart from that it is simplicity itself: a transparent introduction to what all of physics is today: the construction of mathematics that, miraculously, reproduces our observations of the world—and which also successfully predicts the results of future observations—as so magnificently demonstrated at this glorious Symposium!

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1. Introduction

My education poster is the third page of this paper—the rest guide educators in their use of the poster. For the poster is terse—it is the job of the teacher to guide the student and induce her to simply write out the entire poster! For that is the secret of coping with equations. For me, just as for (I think) the typical student, equations, at first, seem simply to bounce off the eyeballs. But, if each equation is actually written out, then the miracle occurs: for to be written out requires each equation to have passed through the eyes; and into the brain; and then, out onto the page—and understanding occurs.

Good students are superb *readers:* just a glance at words in a text conveys meaning. But a glance at an equation is another story! Writing out an equation produces *understanding:* and here, the algebra is trivial, and no steps, however simple, have been left to the reader.

2. The Pythagorean Theorem

The Pythagorean Theorem is known to every educated person, not just in its lyrical statement that "the square on the hypotenuse is equal to the sum of the squares on the other two sides," but also in its algebraic expression, $s^2 = x^2 + y^2$, which of course is what we will need when we will—momentarily—zoom to Einstein's special relativity.

Look at the poster (page 3 of this paper) and see that the first step is the Pythagorean Theorem, here illustrated in such a way that the concept of invariance under a transformation is made clear. The transformation is rotation of the Cartesian coordinates, which clearly should not, and clearly does not, change the length of the hypotenuse.

My poster needs no calculus: simple high school algebra is all that is needed. But I do use one *notation* from calculus: if x, say, is a coordinate, then an interval along the x axis (e.g. $x_2 - x_1$) I denote by $dx \equiv x_2 - x_1$. (While in calculus dx would mean an *infinitesimal* interval in the x direction, in this paper dx means *any* interval in the x direction.)

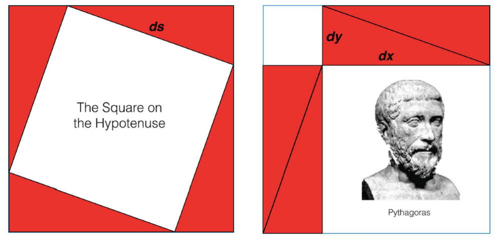

Figure 1. The Theorem of Pythagoras: the two big squares are identical in area. All 8 triangles are identical in area. Equals taken from equals leaves equals, so if you throw out all of the triangles, what remains in each of the two diagrams must be equal. (And: no algebra is used at all!)

I now introduce a *supplement* to my poster to assist the teacher who is guiding her students through the poster. The teacher should ask the student, “how do we know that the Pythagorean Theorem itself is actually true?”

The teacher should then show the student **Figure 1**, which is an instantaneous and (most importantly!) *non-algebraic* proof of the Pythagorean Theorem, revealing that theorem to be a *property of the space in which we live.*

Now, while our figure does include identification of dx, dy, and ds (the hypotenuse), these are *not* involved in the proof. So when the algebra *is* introduced, it *reproduces* reality! That is why algebra was invented. That is also why it is so powerful! Now we, every one of us, are stuck, forever, at the present moment of time. But algebra allows us, intellectually and (it turns out) correctly to *navigate* time—we can extend algebra to predict additional aspects of reality!

3. William Rowan Hamilton

Hamilton, living after Newton but before Einstein, was perhaps the best mathematical physicist who ever lived. But being good at mathematics is not everything, as we shall see! For Hamilton asked a question which perhaps has occurred to your students: the Pythagorean Theorem in three dimensions is clearly $s^2 = x^2 + y^2 + z^2$, and of course, beyond that, it is easy enough to think of time as being a fourth dimension. And so Hamilton examined $s^2 = x^2 + y^2 + z^2 + t^2$ which is what the Pythagorean Theorem would be in four dimensions. But that would be only adding in a fourth (and non-existent) *space* dimension—despite our optimistically labeling it $t$ime. So—Hamilton gave up!

Well, just three years after Einstein’s 1905 discovery of his famous theory of relativity, his former teacher, Hermann Minkowski, discovered that Einstein’s hugely successful “Theory of Relativity” was nothing more than the discovery that in our universe, the separation of two events (for example, two snaps of your fingers, as you wave your arm about) is (brace yourself) $s^2 = x^2 + y^2 + z^2 - t^2$. This realization, by Minkowski, is the most important insight into the nature of the universe in the history of human thought.

And surely poor Hamilton rolled over in his grave: we now see that having imagination is at least as important as mathematical ability in the exploration of the universe!

4. The Statement of Einstein’s Theory of Relativity

$d\alpha^2 + d\beta^2 + d\gamma^2 - d\tau^2 = dx^2 + dy^2 + dz^2 - dt^2$ is Einstein’s theory of relativity, complete (although Einstein himself did not realize that in 1905 when he seeded its discovery!)

The invariance is under a transformation that is quite different from the transformation (rotation) that is illustrated at the top of the poster. No rotation is involved in the transformation of special relativity—just a transformation from a *stationary* coordinate system to a coordinate system that is *moving* (with respect to the first) in a fixed direction and with constant velocity (say, $v = dx/dt$). The simplest transformation imaginable!

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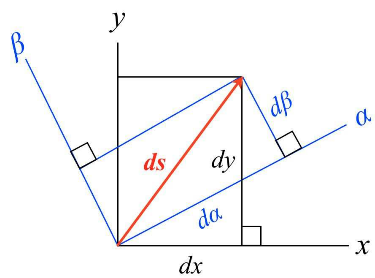
The first step:

We remind ourselves about Pythagorean invariance under a two-dimensional rotation in space:



$$d\alpha^2 + d\beta^2 = ds^2 = dx^2 + dy^2$$

And then—the crucial next step:

Suppose that your sister is moving along your x-axis at a constant speed v

and that during that motion, your sister snaps her fingers twice, $d\tau$ seconds apart —

for you, those two finger-snaps are separated in space by dx and are separated in time by dt

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RELATIVITY: → **Pythagoras, Einstein, and Minkowski assert that $d\tau$ and dt are related by:**

your Sister → $d\alpha^2 + d\beta^2 + d\gamma^2 - d\tau^2 = ds^2 = dx^2 + dy^2 + dz^2 - dt^2$ ← **You**

If so, then $0^2 + 0^2 + 0^2 - d\tau^2 = dx^2 - 0^2 - 0^2 - dt^2$. **And now—let us process this result!**

$d\tau^2 = dt^2 - dx^2$ ← If they are right (and experiment shows they *are*) **time slows down** for your sister!

but $dx = v\,dt$ so $d\tau^2 = (1 - v^2)\,dt^2$ and so $(\frac{dx}{dt} =)$ $v \leqslant \mathbf{1}$! ← A universal **speed-limit** c **= ONE**

For the case $v = 1$, $d\tau = 0$ ← and so *photons never exist!* Next: **E ≡ Energy** and **p ≡ momentum**

$d\tau^2 = dt^2 - dx^2 \rightarrow 1 = (\frac{dt}{d\tau})^2 - (\frac{dx}{d\tau})^2 = \frac{1}{1-v^2} - (\frac{dx}{dt} \cdot \frac{dt}{d\tau} = v\frac{dt}{d\tau})^2$ **or** $\mathbf{1} = \frac{1}{1-v^2} - v^2(\frac{1}{1-v^2})$ **RELATIVITY**

Now multiply through by the square of mass m: $m^2 = m^2(\frac{1}{1-v^2}) - m^2v^2(\frac{1}{1-v^2}) \equiv \mathbf{E^2} - \mathbf{p^2}$

$\gamma \equiv \sqrt{\frac{1}{1-v^2}} = 1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \ldots$ ← from **Newton** [for $v < 0.1$, $1 \leqslant \gamma < 1.00503\cdots$ i.e. $\gamma \approx \mathbf{1}$]

and so: $m^2 = m^2\gamma^2 - (\gamma m v)^2 = m^2(1 + \frac{1}{2}v^2 + \frac{3}{8}v^4 + \cdots)^2 - (\gamma m v)^2$ $\mathbf{E} = \frac{m}{\sqrt{1-v^2}}$ $\mathbf{p} = \frac{mv}{\sqrt{1-v^2}}$

$m^2 = m^2[\mathbf{1} + \frac{1}{2}(\frac{v}{c})^2 + \frac{3}{8}(\frac{v}{c})^4 + \cdots]^2 - m^2[\gamma(\frac{v}{c})]^2$ $c = 1$ *or –but only by convention!–* = 299,792,458 m/s

$m^2c^2 = \frac{1}{c^2}[\,m\,c^2 + \frac{1}{2}m\,v^2 + \frac{3}{8}m\,\frac{v^4}{c^2} + \ldots]^2 - [\,\gamma\,m\,v\,]^2$ ← **we spot** Newton's $E = \frac{1}{2}m\,v^2$ **and** his $p = m\,v$!

$\boldsymbol{m^2c^2 = \frac{E^2}{c^2} - p^2}$ and so, **beyond Newton:** → $\mathbf{E = m\,c^2} + \frac{1}{2}m\,v^2 + \cdots$ and $\mathbf{p} = \gamma\,m\,v$

And if $v = 0$ we find that—from the (extended) Pythagorean theorem—we have successfully derived

the most famous equation in human history: $E = m\,c^2$ ... which has given us the Atomic Bomb:

$neutron + {}^{235}U \rightarrow {}^{236}U \rightarrow {}^{92}Kr + {}^{141}Ba + \langle E = [\,m_U - (m_{Kr} + m_{Ba})\,] \times c^2\rangle + 3\,neutrons$ out →

5. Consequences of Einstein's Theory of Relativity

The poster digs into this "extended Pythagorean Theorem" (which, today, is called Einstein's Theory of Relativity): with the help of Newton, we will understand its *meaning.*

Suppose that you are stationary in your coordinate system (x, y, z, t) [t being the time according to your (stationary) clocks] — and *what happens* is that your sister, who is moving with respect to you in a straight line (with constant velocity v) in her moving coordinate system $(\alpha, \beta, \gamma, \tau)$ [with τ being the time as measured by all clocks that are moving with her in her (moving) reference frame], *snaps her fingers twice:* two "events!"

Let your sister's (moving) α axis lie atop your (stationary) x axis, and let the motion take place along that axis. Then we can simply ignore the other two space dimensions.

In your sister's moving reference frame the two finger snaps occur where she is: at locations $(0, 0, 0, \tau_1)$ and $(0, 0, 0, \tau_2)$—and of course have no spatial separation, only a time separation $\tau_2 - \tau_1 \equiv d\tau$; while for you, her two finger snaps have locations$(x_1, 0, 0, t_1)$ and $(x_2, 0, 0, t_2)$, and so have a spacetime separation of $(x_2 - x_1, 0, 0, t_2 - t_1) \equiv (dx, 0, 0, dt)$.

So our relativity equation, for the separation of her two finger-snaps, is $0^2 + 0^2 + 0^2 - d\tau^2 = dx^2 + 0^2 + 0^2 - dt^2$, or simply $d\tau^2 = dt^2 - dx^2$. But the distance you see your sister go in dt is $dx = v\,dt$ and so we get the poster result that $d\tau^2 = (1 - v^2)\,dt^2$. Now any number, positive OR negative, when squared, is positive. But if v were greater than one, the right-hand side of our equation would be negative—and yet required to be equal to the left-hand side which is necessarily positive! We conclude that if Einstein's theory governs the universe in which we live, **velocities greater than 1 are impossible.**

Experiment has shown that this is true. The fastest speed we have ever measured is the speed of light, which is: **one** light-year per year. That is, **1**. No physical units!

[Ah, units! One reason physics is so confusing is is that we humans are country bumpkins: the speed of light is not really $c = 299{,}792{,}458$ meters per second, it is $c = 1$. One! **1**! No units! Put $E = m$ on your T-shirt, not $E = m\,c^2$! [And Planck's constant (quantum mechanics) is not 6.625×10^{-27} erg-seconds: **it is**: yes you guessed it, **1**!] We use the units that we do partly so that we won't have to use extremely large or extremely small numbers in our daily lives. Our daily work takes place in an extremely limited portion of the available universe. In fact we are all hicks in an *inconceivably big* universe that contains things that are *inconceivably small.* For example, inside a billion-solar-mass black hole is a singularity, having that mass, that is much smaller than one single proton!]

More simple processing yields my poster's $m^2 c^2 = \frac{1}{c^2}(m\,c^2 + \frac{1}{2}m\,v^2 + \ldots)^2 - (\gamma\,m\,v)^2$. If a motionless mass m is in front of you the equation says that $m = m$ which is true!

But if you enter a different frame, in which the mass moves at v, our equation says that what is present is not m but is rather the (measurable) quantities $(E \equiv)\,m\,c^2 + \frac{1}{2}m\,v^2 + \ldots$ and $(p \equiv)\,\gamma\,m\,v$, where we can now (joyfully!) recognize Newtonian kinetic energy and Newtonian momentum hiding inside Einstein's results, and we have discovered that with the mass in motion $E = m$ plus, if $v \ll 1$, just a (relatively) *tiny bit more* energy!

(By the way: I hope you noticed that our equation $1 = \frac{1}{1-v^2} - v^2(\frac{1}{1-v^2})$ is an identity!)

A teacher can bring home the importance of all this to her students by pointing out that LHC (Europe's Large Hadron Collider) pumps arbitrarily large amounts of energy into single particles yet despite their huge energy none of those particles ever exceeds $v = c = 1$*!* So *where does their extra kinetic energy reside?* It is in the infinite number of additional terms beyond $\frac{1}{2}m\,v^2$ that I have designated by simply $+\ldots$

I thank Professor Gabriela González for creating this timely Symposium.

References

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