Mass Does **Not** Increase with Velocity

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Conservation Laws are a fundamental feature of the world, whether the world is Newtonian, Relativistic, or Quantum Mechanical: Emmy Noether showed that conservation of linear momentum results from the symmetry of the world under translations through space, and conservation of energy results from the symmetry of the world under translations through time.

As a result, in Newtonian physics,

Linear Momentum $\mathbf{p} = m\left(\frac{dx}{dt}\,\hat{\mathbf{i}} + \frac{dy}{dt}\,\hat{\mathbf{j}} + \frac{dz}{dt}\,\hat{\mathbf{k}}\right)$ is conserved, and

Kinetic Energy $K = \frac{1}{2}m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \frac{1}{2}m\,v^2$ is also conserved.

(Please notice, for future reference, that $p^2 = \mathbf{p}\cdot\mathbf{p} = m^2\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\}$)

But, in fact, neither of these quantities is actually conserved, because Newtonian physics is incorrect physics. The problem is that Newton thought that there is only one time t which is the same for everyone: absolute time. Well, that is not so. In Special Relativity (tested correct physics), time is different, for differently moving observers.

In particular, if the time on *your* wrist watch is t, the time on the wrist watch of *Alice*, as she passes you by at speed

$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$, is a *different* number t' ,which can be found by considering this equation:

$$\textit{You} \rightarrow \qquad c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 \qquad \leftarrow \textit{Alice}$$

where $c \equiv$ 299,792,458 *m/s* (conversion factor from *seconds* to *meters*, but, *also*, is identical to the speed of light), and where $(t',\, x',\, y',\, z')$ and $(t'+dt',\, x'+dx',\, y'+dy',\, z'+dz')$ are the coordinates of two events as recorded by Alice, and $(t,\, x,\, y,\, z)$ and $(t+dt,\, x+dx,\, y+dy,\, z+dz)$ are the coordinates of *those same two events* as recorded by you.

Note that, according to Newton, $dt' = dt$, and our equation is nothing but the Pythagorean Theorem. OUR EQUATION *IS* THE EXTENSION OF THE PYTHAGOREN THEOREM TO INCLUDE TIME AS THE 'FOURTH DIMENSION,' AND *IS* THE FUNDAMENTAL HYPOTHESIS OF SPECIAL RELATIVITY. That the hypothesis is *correct* has been verified using clocks: say the "two events" in question are successive ticks of Alice's watch, in which case our equation becomes $c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - 0^2 - 0^2 - 0^2$ (since both ticks occur at the *same* location for Alice). The two ticks are separated in time by $dt' \equiv d\tau$ for Alice, and by dt for you. (Since the clock is *not moving* in Alice's frame [and *only* in Alice's frame], $t' \equiv \tau$ is called the *proper time*.)

$$c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

$$c^2\left(\frac{d\tau}{dt}\right)^2 = c^2 - \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = c^2 - v^2 \qquad c^2d\tau^2 = dt^2\left(c^2 - v^2\right)$$

$$dt = \frac{c\,d\tau}{\sqrt{c^2 - v^2}} \qquad \boxed{dt = \frac{d\tau}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Notice that Alice cannot go past you at $v > c$, or our equation would say that a real number dt is equal to an imaginary number, which cannot be true. THIS IS THE *ONLY* REASON WHY NOTHING CAN GO FASTER THAN LIGHT. Notice that **mass** *has not appeared at all* in our relativistic work so far. Notice, also, that we have predicted *time dilation:* if Alice is passing you at, say, half the speed of light, then $dt = 1.155\,d\tau$, and Alice's watch is ticking at a slower rate than your own watch. EXPERIMENT SHOWS THIS TO BE TRUE, thus *validating* the fundamental hypothesis of Special Relativity.

Now multiply our equation by the square of Alice's mass:

$$m^2c^2d\tau^2 = m^2c^2dt^2 - m^2\left(dx^2 + dy^2 + dz^2\right)$$

$$m^2c^2 = m^2c^2\left(\frac{dt}{d\tau}\right)^2 - m^2\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right\}$$

$$m^2c^2 = \left\{mc\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right\}^2 - m^2\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right\}$$

$$m^2c^2 = \left\{\frac{mc^2}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)\right\}^2 - m^2\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right\}$$

$$m^2c^2 = \frac{1}{c^2}\left(mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \cdots\right)^2 - m^2\left\{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2\right\}$$

We recognize that Newton's kinetic energy has *appeared* in our equation, and also something else has appeared that *would be* the square of Newton's momentum, if there really *were* only one absolute time $(\tau = t)$. (Please compare with that equation that we saved "for future reference.")

Our equation becomes $m^2c^2 = \frac{E^2}{c^2} - p^2$ or $\boxed{E^2 = (pc)^2 + (mc^2)^2}$ if we make two new ***definitions:***

Total energy $E \equiv mc^2 + K = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}mv^2\left(\frac{v^2}{c^2}\right) + \cdots$

and **momentum** $\mathbf{p} \equiv m\left\{\left(\frac{dx}{d\tau}\right)\hat{\mathbf{i}} + \left(\frac{dx}{d\tau}\right)\hat{\mathbf{j}} + \left(\frac{dx}{d\tau}\right)\hat{\mathbf{k}}\right\}$

} Einstein (correct)

Please notice that if Alice is *not moving* $($*i.e.*, if $K = 0)$, our definition says that her total energy is $E = mc^2$, the most famous equation in physics. Our definitions *are* appropriate, and these *are* the quantities that *are* conserved: *not* Newton's quantities, which were, recall:

Total energy = kinetic energy, $K = \frac{1}{2}mv^2$

And momentum $\mathbf{p} = m\left\{\left(\frac{dx}{dt}\right)\hat{\mathbf{i}} + \left(\frac{dx}{dt}\right)\hat{\mathbf{j}} + \left(\frac{dx}{dt}\right)\hat{\mathbf{k}}\right\}$

} Newton (incorrect)

The differences are small, between the correct relativistic values for kinetic energy and momentum, and the incorrect Newtonian values, if Alice's velocity is much smaller than the velocity of light.

We are done! And you will notice that we have successfully treated mass m as an *invariant*: the same as measured by you, *and* as measured by Alice.

But why, then, does the famous Wolfgang Rindler (on page 98 of his book, "Essential Relativity") say that "the inertial mass of a particle increases with v from a minimum of m_0 at $v = 0$ to infinity as $v \to c$?"

Rindler comes to this incorrect conclusion by defining a vector $p \equiv (mv_x, mv_y, mv_z)$ [which is *not* conserved; it is Newtonian momentum] and saying "assume this is conserved." Then [in order to make this be, in fact, actually conserved] Rindler arbitrarily sets

$m = \gamma(v) m_0$ [Rindler's equation 50.3], where $\gamma(v) \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

Let's see that this legerdemain *does in fact work:*

$$\mathbf{p} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dx}{dt}\hat{\mathbf{i}} + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dy}{dt}\hat{\mathbf{j}} + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dz}{dt}\hat{\mathbf{k}}$$

but we know that $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{dt}{d\tau}$, so we get

$$\mathbf{p} = m_0\frac{dt}{d\tau}\frac{dx}{dt}\hat{\mathbf{i}} + m_0\frac{dt}{d\tau}\frac{dy}{dt}\hat{\mathbf{j}} + m_0\frac{dt}{d\tau}\frac{dz}{dt}\hat{\mathbf{k}} = m_0\left(\frac{dx}{d\tau}\hat{\mathbf{i}} + \frac{dy}{d\tau}\hat{\mathbf{j}} + \frac{dz}{d\tau}\hat{\mathbf{k}}\right) = \text{Einstein's } \mathbf{p}$$

which *is indeed conserved*, with m_0 being our invariant mass m.

But we know that Rindler's association of the γ with mass is incorrect: we saw earlier that γ is associated with *time*, not *mass*, as verified by experiments that *do not involve mass at all!*

Why does this foolish association of γ with mass persist, even amongst some Professors?

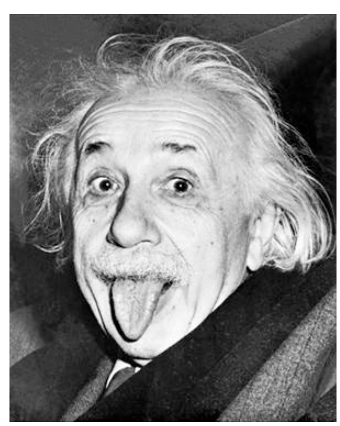
Perhaps, it is a nostalgic desire to keep what at least *looks like* Newton's incorrect expression for the momentum? If so, how sad.

I suspect that a second reason this error persists, is that it *does seem* to provide a "logical" explanation for why nothing can go faster than light. Indeed, Rindler says "We should not be too surprised at this, since there must be *some* process in nature to prevent particles from being accelerated beyond the speed of light." *Oy!*

But, above, we have already *understood* why nothing can go faster than light: doing so is incompatible with the (relativistic) Pythagorean geometry of Spacetime.

MASS HAS NOTHING TO DO WITH IT AT ALL!

In the frame of *any* photon, the Earth is moving at the speed of light. If mass increased with velocity, the Earth would be infinitely massive … and would collapse into a Black Hole. Well, *Earth does not!*



Alice has **Two** Velocities

We are not accustomed to thinking in terms of *two* velocities, simply because we are not accustomed to thinking in terms of two *times*.

Let's go back to our equation

$$m^2c^2 = \left\{ mc\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \right\}^2 - m^2\left\{ \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 \right\}$$

Notice that we can cancel the m^2's. (Hey, *we* put them in—*we* can take them out!) Nothing is lost by having Alice moving *only* in the *x* direction. And, instead of setting $c = 299{,}792{,}458$ (merely an historical curiosity), let us set $c = 1$.

Alice's speed $v = \frac{dx}{dt}$ cannot exceed unity! A much better way to say it!

Anyway, our equation now becomes, since Alice is now moving only in the *x* direction:

$$1 = \left(1-v^2\right)^{-1} - u_x^2$$

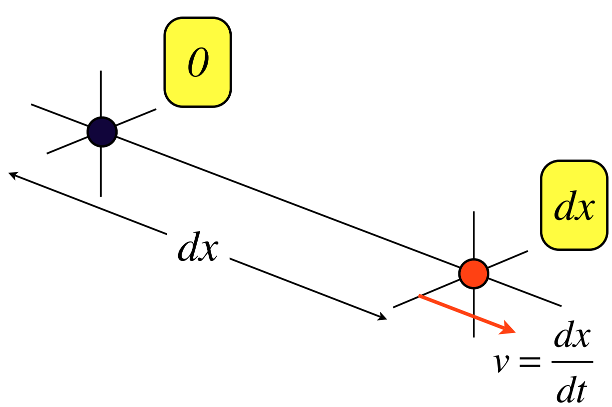
where $u_x \equiv \frac{dx}{d\tau}$ and so $1+u_x^2 = \left(1-v^2\right)^{-1} = 1+v^2+v^4+v^6+v^8+v^{10}+v^{12}+\cdots$

and so $u_x^2 = v^2+v^4+v^6+v^8+v^{10}+v^{12}+\cdots = \dfrac{1}{\frac{1}{v^2}-1} = \dfrac{v^2}{1-v^2}$ $\qquad v^2 = \dfrac{u_x^2}{1+u_x^2}$

or $u_x^2 = v^2\left(1+v^2\left(1+v^2\left(1+v^2\left(1+v^2\left(1+v^2\cdots\right.\right.\right.\right.\right.$ v can only go to 1; u_x can go to infinity!

Also, $v^2 = u_x^2(1-u_x^2(1-u_x^2(1-u_x^2(-u_x^2(1-u_x^2\cdots$

It is easy to understand the difference between the two velocities. Notice the two yellow mileposts. You are the black circle; Alice is the red circle. To get *v,* use *your* wrist watch; to get *u,* use *Alice's* wrist watch. The *dx* is the same!



To Alice, her own momentum is zero, and her energy is m

To you, her momentum is $p_x = m\frac{dx}{d\tau}$,

and her energy is

$$E = m\left\{1+\frac{1}{2}\left(\frac{dx}{dt}\right)^2+\frac{3}{8}\left(\frac{dx}{dt}\right)^4+\cdots\right\}$$

I still don't know what Alice's "second velocity" actually *means!*

Well, *let's find out* what it means!

We all live on the outskirts of our galaxy—a galaxy that is about 100,000 light years in diameter. Which means that a photon of light from your flashlight, shined at night in the right direction, will arrive at the other side of the galaxy 100,000 years from now (by your watch).

Suppose, just as you send off your photon, Alice's spacecraft goes by (in that same direction) at almost the speed of light: let's say, $v = 0.9999999c$. Then Alice will arrive at the other side of the galaxy a few days after your photon arrives (by *your* watch).

Hey, let's check that: $\left(100{,}000 - 100{,}000 \times 0.9999999\right) \times 365.2422 = 3.65\,days$ Ta da!

As Alice arrives at the other side of the galaxy 100,000 years have passed for you. How much time has passed *for Alice*?

$$u_x = \frac{dx}{d\tau} = \frac{1}{\sqrt{\frac{1}{v^2} - 1}} = \frac{1}{\sqrt{\frac{1}{\left(0.9999999\right)^2} - 1}} = 2236.07 \quad \text{so}$$

$$d\tau = \frac{dx}{2236} = \frac{100{,}000}{2236} = 45\,years$$

So for Alice, just 45 years have passed when she finds that she has arrived at the other side of the galaxy. Alice would say, with some justice, that she has traveled at *2236 times the speed of light.* (Of course that photon you sent would laugh to hear that, and would claim that, if so, then *it* had gone *infinitely faster* than light!)

The interesting thing is the time dilation. The "second velocity" adds nothing to that—it is merely *reflecting* that.

$$p = m u_x = \frac{m v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E = m c^2 \sqrt{1 + \frac{u_x^2}{c^2}} = \frac{m c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

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