Bell's Theorem, Quantum Theory and Conceptions of the Universe
Ed. M. Kafatos ©1989 Kluwer Academic Publishers

TEACHING QM: TRUE, TRIVIAL, INEVITABLE

RICHARD CONN HENRY

Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218 USA

Abstract. Quantum Mechanics can (and should) be taught so as to bring out the utter simplicity of its origin, and its inevitable character. There are mysteries in the universe, but QM is not one of them.

1. INTRODUCTION

Bell's Theorem, EPR, wave-particle duality, many-worlds ... what fun; but, as we shall see, what bosh. There are, however, genuine mysteries in the world. Let me specify two of them:

1.1. First Mystery

Pythagoras realized that the theorem that bears his name says something profound and mysterious about the world: distances are real numbers. He generalized: "number is all things." He was right. Consider any measurement: you read it on a scale, laid out in one dimension of space. Real numbers. Contemplating QM, Eddington said "the stuff of the world is mind stuff." Pythagoras knew that!

1.2. Second Mystery

The Hamiltonian is invariant under spatial translations, temporal translations, and spatial rotations. The results of these facts are conservation of linear momentum, energy, and angular momentum, respectively. This has nothing to do with QM: these results are present in classical mechanics too, and they amount to definitions of the words momentum and energy. That they are profound properties of the universe is shown most joyously by the slight violation of a similar symmetry, parity conservation, in the weak interaction.

2. QUANTUM MECHANICS

Keeping the two mysteries in mind, let us make plans to observe the universe, that is, to make some measurements. What will we measure? a) positions, because they are natural to measure, and because Pythagoras has told us what the measurements (eigenvalues) will be: any real number; and b) momentum, quite unnatural to measure (there was great debate at the time of Newton as to what was the measure of "quantity of motion"), and energy; because they are conserved.

We will discover as we plan our observations that we automatically produce Quantum Mechanics. It is not an option. Let us begin!

2.1. Predict the Result of your Observation, Knowing Nothing

You can do this, better than you think. Plot the probability of getting a particular result, against the value of the result (the eigenvalue). The probability must lie between 0 and 1; the possible eigenvalues are the real numbers (Pythagoras tells us). Your plotted curve must be dashed, because its shape is unknown.

Not bad, considering we knew nothing!

2.2. Represent that Unknown Curve in a Much Better Way

There is additional information that we haven't used yet, and that is, that you get a definite result when you make any measurement. (You know that from experience! Once you've made a measurement, there is zero probability that the measured quantity is not what you have just measured.) Your probability curve does not incorporate this fact. So supplement it by constructing an infinite real orthogonal Hilbert-style space. Infinite because there are infinitely many possible eigenvalues, the real numbers. Orthogonal because they are independent values. Real, because that is natural.

Your "dashed curve" appears in this space as a vector. The vector's projection on the infinitely-many axes (the eigenvectors) are the infinitely many points on your dashed curve. As you waggle the state vector around in the space, the dashed curve waggles around, the integral under it remaining unity.

What could be simpler!

2.3. Consider Similar Plans for Measurement of Some Other Quantity

The result of course will be exactly the same.

Now pick up the second Hilbert space and plunk it down, origin-on-origin, on the first Hilbert space, and then rotate one (or the other, or both) so that the two state vectors coincide. (Since we don't know where either state vector is located, this must remain a purely mental exercise).

What is the result? In particular, do the eigenvectors of the first Hilbert space lie on those of the second Hilbert space? If they all do, then when you make a measurement of one of the quantities, the system will be in a state where you can expect a specific definite result when you measure the other quantity. If none of them do, then when you make a measurement of one of the quantities, the system will be in a state where only the probability of any specific value of the other quantity can be predicted; that is, if the quantities are position and momentum, you will have the uncertainty relation.

2.4. So Tell Me, What is the Result?

We can't find out from our Hilbert spaces by using geometrical methods, because it's too hard to visualize. So associate an infinite matrix with each Hilbert space (each "operator"). The matrix elements depend on the basis chosen, but in its own basis, each matrix is simply the unit matrix. However, cleverly put the eigenvalues down the diagonal, instead of just unity. The result is obviously $\Omega\,|\psi\rangle = \omega\,|\psi\rangle$, where Ω is the operator (matrix) and ω the eigenvalue. We are now armed for bear!

2.5. A Big Surprise

We do know the position eigenvalues, thanks to Pythagoras, so let's work in the position basis. We want to solve the momentum eigenvalue equation, $P\,|p\rangle = p\,|p\rangle$. We don't yet know what to use for P, but before we even start to track that down, we get a terrific surprise, because we recognize that to get all real eigenvalues (as required to keep Pythagoras happy!), we must (vaguely paradoxically) have a complex Hilbert space! So we have to introduce complex probability amplitudes, which (squared) give the (necessarily real) probabilities.

2.6. Momentum and Energy

Let $T = I + \varepsilon K$ acting on the state vector cause infinitesimal translation (Second Mystery) of the system through space. Then

$$\langle x|\,(I+\varepsilon K)\,|\psi\rangle = \langle x|\,\psi'\rangle = \psi'\,(x) = \psi\,(x) + \zeta\frac{d\psi}{dx}$$

where in the final step the translated wave function is expanded in a Taylor series. So

$$\langle x|\,K\,|\psi\rangle = \tfrac{\zeta}{\varepsilon}\tfrac{d\psi}{dx} \quad \text{, and let } \tfrac{\zeta}{\varepsilon} \equiv \hbar$$

Result is $\quad \langle x|\,P\,|\psi\rangle = -i\hbar\tfrac{d}{dx}\,\langle x|\,\psi\rangle$

and if $I+\varepsilon L$ causes infinitesimal translation through time, similarly,

$$H\,|\psi\rangle = i\hbar\left|\dot{\psi}\right\rangle$$

where $iP = K$ and $H/i = L$ are introduced to make P and H Hermitian, and hence make all our eigenvalues real (First Mystery).

3. CONCLUSION

We have just produced QM (our last two equations) from essentially nothing, beyond our First and Second Mysteries. What QM asserts (no matter how bizarre, e.g. EPR) must be true; experiment is hardly needed. This work was supported by National Aeronautics and Space Administration grant NAG5-619.

©1989 Kluwer Academic Publishers

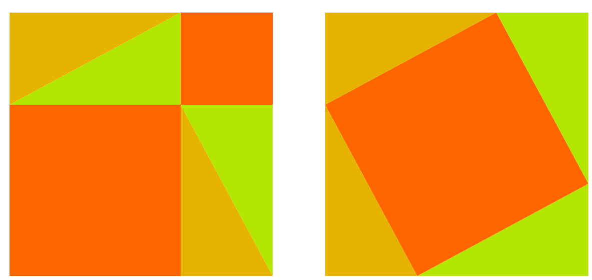

Fig. 1. This paper has been reTeXed in 2004 September, and this figure (by Harald Hanche-Olsen) added in tribute to Pythagoras.