Quantum Mechanics is Vitally-Important! And happily, QM is Very Understandable!

Richard Conn Henry

This (two-page-long) paper is a concise summary of my lengthy, mathematically-detailed, paper “Quantum Mechanics Made Transparent” (American Journal of Physics 58, 1087, 1990).

Pythagoras in ancient times, proclaimed his famous Theorem: That for a right-angled triangle, “the square on the hypotenuse is equal to the sum of the squares on the other two sides”. That this is, in fact, true, was proved by Euclid, but, it was only in 1939 that a Belgian-American high-school student, Maurice Laisnai, found the astoundingly SIMPLE proof that Pythagoras was correct. In that spirit, here is my own, utterly SIMPLE, derivation of Quantum Mechanics.

Pythagoras also gave us a critical mystery, in claiming “Number is all things”! He was right! Conservation of energy, conservation of momentum, and conservation of angular momentum: all these were postulated, and all survived experimental test. But: WHY conservation? It was an American mathematician, Emmy Noether, who DISCOVERED why. Emmy recognized that spacetime (the universe) has simple (and unsurprising) symmetries. Emmy discovered that the reason that linear momentum is conserved is that the universe is symmetric under translation through space. (You know that! Move your experiment sideways, and you find that you get the same result!) Emmy showed by simple math that this fact results in the conservation of linear momentum. And, Emmy also found: rotate your experiment, and you get the same result (but rotated), so angular momentum is conserved! And Emmy also found: repeat your experiment an hour later, and you get the same result. So, energy is conserved! (While Emmy discovered the vital concept, it was Eugene Wigner who applied it to physics. He won the Nobel prize!)

In 1925, quantum mechanics was discovered, by Werner Heisenberg and Erwin Schrödinger. Quantum mechanics TOO (as I will now show you) is produced, in totally straightforward fashion, from those identical symmetries.

Isaac Newton (and his successors) did produce a hugely successful mechanics: \( F = m a \), etc, which does produce tremendous engineering success. However, at the level of molecules and atoms it all breaks down totally. So, WE will start completely afresh: we will entirely bypass dear old Newton! We, my friends, are going to make some measurements of the world, and we are going to assume absolutely nothing! We will discover quantum mechanics automatically.

Let us start with position, because Pythagoras tells us that our measurement will be a number. It could be any number, from minus-infinity, to plus-infinity. The probability of it being any particular number is unknown, but of course it must lie between zero and one. Let’s make a plot of probability of a particular result, against all those numbers from minus-infinity to plus-infinity. Our curve must be dashed, of course, since we don’t yet know the actual probabilities.

We do also know that we get a definite result when we make any measurement. (We know that from experience: once we have made a measurement, there is zero probability that the measured quantity is other than what we have just measured.) But our probability curve does not yet incorporate this fact. So supplement it by constructing a flat, orthogonal, infinite-dimensional, space. Infinite because there are infinitely many possibilities. Orthogonal, because they are all independent values. We will make it real, because at this point, that seems natural. We’ll call it a Hilbert space. Our “dashed curve” appears in Hilbert space as a vector, the “state vector”. That vector’s projection on the infinitely-many axes are the infinitely many points on our dashed curve. As you waggle the state vector around in Hilbert space, the dashed curve waggles about: BUT, now, the integral under it must remain UNITY! What could be simpler?
Next, we will ask the same question, but now for SOME OTHER measurable quantity, such as, for example **momentum**. The result (at this point) will be exactly the same; we are exploring!

Now pick up that second Hilbert space, and plunk it down, origin-on-origin, on top of the first Hilbert space, and then, rotate one, or both, until the two state vectors coincide.

What is the result? In particular, do all the axes of the first Hilbert space lie atop those of the second Hilbert space? If they all do, then when you make a measurement of one of the quantities, the system will be in a state where you must get a specific definite result when you measure the **other** quantity. If none of them do, then when you make a measurement of **one** of the two quantities, the system will be in a state where only the probability of any specific value of the **other** quantity can be predicted; that is, if the quantities are position and momentum, you will have the famous uncertainty relation of quantum mechanics!

So, what is the result for our two quantities? We can’t find out from our Hilbert spaces by using geometrical methods, because it’s too hard to visualize. So instead, associate an infinite matrix with each Hilbert space (each “operator”). The matrix elements depend on the basis chosen, but in its own basis, each matrix is simply a unit matrix. However, **now, critically, we** put the possible outcomes (the “eigenvalues”) down the diagonal, instead of just unity. The result is obviously \( \Omega |\psi\rangle = \omega |\psi\rangle \) where \( \Omega \) is the operator (matrix), and \( \omega \) is the eigenvalue. (Those funny brackets are Dirac’s notation for an infinite column vector.)

We do know the position eigenvalues (thanks to Pythagoras! ) so let’s work in the position basis. We want to solve the momentum eigenvalue equation, \( P|p\rangle = p|p\rangle \). We don’t, yet, know what to use for \( P \), but, before we even start to try to track that down, we get a terrific surprise, because we recognize that, to get all real eigenvalues (as required to keep Pythagoras happy), we must (vaguely paradoxically) have a complex Hilbert space! So, we must introduce complex probability amplitudes which (when squared) will give necessarily-real probabilities.

Let \( T = I + \varepsilon K \) acting on the state vector cause infinitesimal translation of the system through space. Then

\[
\langle x | (I + \varepsilon K) | \psi \rangle = \langle x | \psi' \rangle = \psi'(x) = \psi(x) + \zeta \frac{d\psi}{dx}
\]

where in the final step the translated wave function is expanded in a Taylor series.

So \( \langle x | K | \psi \rangle = \frac{\zeta}{\varepsilon} \frac{d\psi}{dx} \) and let \( \frac{\zeta}{\varepsilon} \equiv \hbar \) The result is

\[
\langle x | P | \psi \rangle = -i\hbar \frac{d}{dx} \langle x | \psi \rangle
\]

And if \( I + \varepsilon L \) causes infinitesimal translation of the system through time, then, similarly,

\[
H | \psi \rangle = i\hbar | \dot{\psi} \rangle
\]

Where \( iP = K \) and \( H/i = L \) have been introduced in order to make \( P \) and \( H \) Hermitian, and hence make all our possible measurements real (once again, to keep dear Pythagoras happy! ).

**Our two equations** are: 1) the **Momentum Operator**

and, 2) the **Shrödinger Equation**

This is presented, in gruesome detail, in the American Journal of Physics, 58, 1087, 1990