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Derivation of the relativistic 'proper-time' quantum evolution equations from canonical invariance

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Abstract

Based on (1) the spectral resolution of the energy operator; (2) the linearity of correspondence between physical observables and quantum self-adjoint operators; (3) the definition of conjugate coordinate–momentum variables in classical mechanics; and (4) the fact that the *physical* point in phase space remains unchanged under (canonical) transformations between one pair of conjugate variables to another, we are able to show that $\langle t_s|E_s\rangle$, the proper-time rest-energy transformation matrices, are given as $a\exp(-\mathrm{i}E_s t_s/\hbar)$, from which we obtain the proper-time rest-energy evolution equation $\mathrm{i}\hbar\frac{\partial}{\partial t_s}|\Psi\rangle = \hat{E}_s|\Psi\rangle$. For special relativistic situations this equation can be reduced to the usual $\mathrm{i}\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{E}|\Psi\rangle$ dynamical equations, where t is the 'reference time' and E is the total energy. Extension of these equations to *accelerating* frames is then provided.

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1. Introduction

Attempts at proving the dynamical equations of quantum mechanics (especially the time-dependent Schrödinger equation), rather than *assuming* them [1], as did Schrödinger himself [2], or *postulating* the $H \to \mathrm{i}\hbar\partial/\partial t$ correspondence [3, 4], abound in the literature. Most of these approaches achieve this purpose by showing that the dynamical equations are equivalent to other assumed postulates, such as stochastic dynamics [5–7], path-integrals [8–10] or Galilean symmetry [11]. Some [12] attempt to treat time within the framework of the *time-independent* Schrödinger equation, as a semiclassical quantity associated with the dynamics of a large semiclassical bath coupled to the quantum system of interest. This approach neglects the vast literature and experimental evidence [13] regarding the temporal evolution of wave

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packets in an *isolated* system which cannot be described by the time-independent Schrödinger equation alone.

Recently [14] we have shown that the $[\hat{q}_l, \hat{p}_l] = \mathrm{i}\hbar$ commutation relations between any generalized coordinate operator $\hat{q}_l$ and its conjugate momentum operator $\hat{p}_l$ can be derived from the very definition of conjugate variables in classical mechanics. The derivation made use of the invariance of the free particles' kinetic energy and other invariant operators to (canonical) transformations between one pair of conjugate variables and another. In the present paper we extend this approach to proving the relativistic (and non-relativistic)

$$\mathrm{i}\hbar \frac{\partial}{\partial t_s}|\Psi\rangle = \hat{E}_s|\Psi\rangle \tag{1}$$

quantum evolution equations, with t_s being the proper time and $\hat{E}_s$ the rest-energy operator.

In the coordinate–momentum case the $\hat{p} = -\mathrm{i}\hbar\partial/\partial x$ equation analogous to equation (1) is equivalent to the $[x, p] = \mathrm{i}\hbar$ commutation relation. This cannot be easily done in the time–energy case because of our seeming inability to construct a self-adjoint operator for the 'time'. In particular, it was claimed by Pauli [15], that if such an operator existed it would imply that its conjugate operator, namely the Hamiltonian, would have a purely continuous spectrum and be unbounded from below. More specifically, if there exists a self-adjoint time operator $\hat{t}$ conjugate to the Hamiltonian $\hat{H}$ operator, such that $[\hat{t}, \hat{H}] = -\mathrm{i}\hbar$, then if $|\Phi_{E_i}\rangle$ is an eigenstate of $\hat{H}$ satisfying $\hat{H}|\Phi_{E_i}\rangle = E_i|\Phi_{E_i}\rangle$, $|\Phi_{E_i-\beta}\rangle \equiv \exp(\mathrm{i}\beta\hat{t}/\hbar)|\Phi_{E_i}\rangle$ would also be an eigenstate of $\hat{H}$ with an eigenvalue $E_i - \beta$. Since β is arbitrary this would imply that the spectrum of $\hat{H}$ is continuous and unbounded from below.

Subsequently it was shown [16] that Pauli's arguments were flawed and that for a bounded time operator a conjugate Hamiltonian with a point spectrum *can* exist. A simplified way of explaining this flaw is to say that if $\langle x|\Phi_{E_i}\rangle$ is square-integrable, for an arbitrary $E_i - \beta$, $\langle x|\Phi_{E_i-\beta}\rangle$ is not square-integrable as it diverges either as $x \to \infty$ or as $x \to -\infty$. It is therefore not part of the spectrum of the Hamiltonian.

In what follows we define a *bounded*, *self-adjoint* 'proper-time' operator, based on the classical 'proper time' in different inertial frames. This operator assumes the form $\hat{t}_s \equiv t\left(1 - \hat{v}_s^2/c^2\right)^{\frac{1}{2}}$, where t is a *parameter* representing the time measured in one of the frames, designated the 'reference' frame, and $\hat{v}_s$ is the velocity operator of the s-frame. We argue that in quantum mechanics there is a natural spread in the observed t_s values due to the uncertainty in the s-frame velocity.

After identifying below the variable conjugate to t_s in relativistic mechanics as $-E_s$, where E_s is the rest energy of particle s, we make use of canonical invariance to prove the $\langle t_s|E_s\rangle = a\exp(-\mathrm{i}E_s t_s/\hbar)$ identity, a special case of which being $\langle t|E\rangle = a\exp(-\mathrm{i}Et/\hbar)$, where E is the total energy, leading to equation (1). Thus, together with the *'ab initio'* derivation of the coordinate representation of the momentum operator presented in [14], we now have a more firmly based, reduced number of axioms, theory of relativistic quantum evolution.

2. Review of the proof of the Cartesian $[\hat{x}, \hat{p}]$ commutation relations in relativistic quantum mechanics

In order to motivate what follows we briefly review part of [14] in which the Cartesian $[\hat{x}, \hat{p}]$ commutation relations of relativistic quantum mechanics were derived. We consider two free particles A and B of equal rest mass $m_A = m_B$ whose Cartesian extensions on the x-axis are denoted as x_A and x_B. The momenta conjugate to these coordinates are obtained from the free Lagrangian [19]

$$\mathcal{L} = -mc^2\left[\left(1-\beta_A^2\right)^{\frac{1}{2}} + \left(1-\beta_B^2\right)^{\frac{1}{2}}\right] \tag{2}$$

where $\beta_{A(B)} = v_{A(B)}/c$, as

$$p_s = \partial\mathcal{L}/\partial v_s = mv_s\big/\left(1-\beta_s^2\right)^{\frac{1}{2}}, \qquad s = A, B. \tag{3}$$

We now make a canonical transformation to the $X_{1(2)} = (x_A \pm x_B)/\sqrt{2}$, variables, with the associated velocities $V_{1(2)} = \dot{X}_{1(2)}$, being given as, $V_{1(2)} = (v_A \pm v_B)/\sqrt{2}$. Substituting the velocity relations into equation (2) we have that

$$-\mathcal{L}/(mc) = \left(c^2 - \frac{(V_1+V_2)^2}{2}\right)^{\frac{1}{2}} + \left(c^2 - \frac{(V_1-V_2)^2}{2}\right)^{\frac{1}{2}}, \tag{4}$$

from which we obtain that the momenta conjugate to X_1 and X_2 are given as

$$P_1 = \frac{\partial\mathcal{L}}{\partial V_1} = \frac{p_A + p_B}{\sqrt{2}}, \qquad P_2 = \frac{\partial\mathcal{L}}{\partial V_2} = \frac{p_A - p_B}{\sqrt{2}}. \tag{5}$$

We now consider the sum of the *squares* of the energies of the two particles,

$$\mathcal{T} = T_A^2 + T_B^2 = c^2\left\{2m^2c^2 + p_A^2 + p_B^2\right\}. \tag{6}$$

It is easy to show that $\mathcal{T}$ is invariant to the canonical transformation to the X_1 and X_2 variables and can be written as $\mathcal{T} = c^2\left\{2m^2c^2 + P_1^2 + P_2^2\right\}$.

Since $\hat{\mathcal{T}}$, the quantum operator corresponding to $\mathcal{T}$, is a function of the $\hat{p}_A$ and $\hat{p}_B$ operators, which commute between themselves, and is also a function of the $\hat{P}_1$ and $\hat{P}_2$ operators, which also commute between themselves, it must commute with all four momenta,

$$[\hat{\mathcal{T}}, \hat{p}_A] = [\hat{\mathcal{T}}, \hat{p}_B] = [\hat{\mathcal{T}}, \hat{P}_1] = [\hat{\mathcal{T}}, \hat{P}_2] = 0.$$

This fact, plus the separable form of $\hat{\mathcal{T}}$, allows us to write the eigenstates of $\hat{\mathcal{T}}$ in two different ways

$$\langle x_A|p_A\rangle\langle x_B|p_B\rangle = \langle X_1|P_1\rangle\langle X_2|P_2\rangle. \tag{7}$$

Choosing two particular momenta, $p_A = p$ and $p_B = -p$, and two particular positions $x_A = x$ and $x_B = -x$, we have for these values that $P_1 = 0$, $P_2 = \sqrt{2}p$, $X_1 = 0$, $X_2 = \sqrt{2}x$. Equation (7) now assumes the special form,

$$\langle x|p\rangle\langle -x|-p\rangle = \langle 0|0\rangle\langle\sqrt{2}x|\sqrt{2}p\rangle. \tag{8}$$

Obviously the $\langle -x|-p\rangle$ amplitude is independent of the definition of our coordinate system. Thus, if we re-define $-x$ to be x, forcing by equation (3) (since the Lagrangian remains invariant to this re-definition), $-p \to p$, we obtain that $\langle -x|-p\rangle = \langle x|p\rangle$, and it follows from equation (8) that

$$\langle 0|0\rangle\langle\sqrt{2}x|\sqrt{2}p\rangle = (\langle x|p\rangle)^2, \quad \text{or that,} \quad \langle\sqrt{2}x|\sqrt{2}p\rangle' = (\langle x|p\rangle')^2, \tag{9}$$

where $\langle x|p\rangle' \equiv \langle x|p\rangle/\langle 0|0\rangle$.

Defining $\delta_p \equiv p/2^{n/2}$, and $\delta_x \equiv x/2^{n/2}$, we have from equation (9) that

$$\langle x|p\rangle' = \langle 2^{n/2}\delta_x|2^{n/2}\delta_p\rangle' = (\langle\delta_x|\delta_p\rangle')^{2^n}.$$

Hence,

$$\log\langle x|p\rangle' = 2^{n/2}\delta_x 2^{n/2}\delta_p\alpha'(\delta_x, \delta_p) = px\alpha'(\delta_x, \delta_p),$$

where

$$\alpha'(\delta_x, \delta_p) = \log\langle\delta_x|\delta_p\rangle'/(\delta_p\delta_x).$$

For every p and x values we can find an n value, such that δ_x and δ_p are sufficiently small so that $\alpha'(\delta_x, \delta_p)$ is sufficiently close to its limiting value

$$\alpha = \lim_{\delta_p\to 0, \delta_x\to 0} \frac{\log\langle\delta_x|\delta_p\rangle'}{\delta_x\delta_p} = \left.\frac{\partial^2 \log\langle x|p\rangle'}{\partial x\partial p}\right|_{x,p=0}.$$

Hence

$$\log(\langle x|p\rangle') = \alpha p x \quad \text{or} \quad \langle x|p\rangle = \langle 0|0\rangle \exp(\alpha p x).$$

In order for $\langle x|p\rangle$ to be normalizable to $\delta(x - x')$, namely

$$\int\int \mathrm{d}p\, \mathrm{d}x' \langle x|p\rangle\langle p|x'\rangle = 1, \tag{10}$$

α must be a purely imaginary number $\alpha = \mathrm{i}\gamma$, (otherwise $\langle x|p\rangle$ would diverge, either as $x \to \infty$ or as $x \to -\infty$). By identifying γ with $1/\hbar$, we obtain that

$$\langle x|p\rangle = \langle 0|0\rangle \exp(\mathrm{i}\gamma p x) = (2\pi\hbar)^{-1/2} \exp(\mathrm{i}px/\hbar) \tag{11}$$

where the identification of the normalization factor $\langle 0|0\rangle$ as $(2\pi\hbar)^{-1/2}$ stems from equation (10). Using equation (11) it is easy to show [3, 20] that

$$\langle x'|\hat{p}|x\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial x}\delta(x - x'), \tag{12}$$

and that

$$[\hat{x}, \hat{p}] = \mathrm{i}\hbar.$$

3. The time–energy transformation and the quantum evolution equation

We now wish to extend the proof presented in the previous section to the time–energy domain. In classical mechanics the use of canonical transformations to treat the time variable as a coordinate is well established [21]. As discussed above, the situation in quantum mechanics is not so straightforward because one has to be careful about the type of time that can be treated as an operator. We now show that there are no problems associated with treating the *'proper time'* as a coordinate and quantizing it.

For an inertial frame, in which the velocity is constant, the classical 'proper time' is given[1] as

$$t_s = t\left(1 - v_s^2/c^2\right)^{\frac{1}{2}}, \tag{13}$$

where t is the time in (an arbitrarily chosen) 'reference' Lorentz frame and $v_s \equiv \mathrm{d}x_s/\mathrm{d}t$ is the velocity of the s-frame *relative* to the reference frame. Alternatively we could work with the Lorentz transformed time

$$t_s = (t - x_s v_s/c^2)/\left(1 - v_s^2/c^2\right)^{\frac{1}{2}}. \tag{14}$$

We can define a self-adjoint bounded 'proper-time' operator by replacing v_s in equation (13) by the velocity operator $\hat{v}_s$. Thus,

$$\hat{t}_s = t\left(1 - \hat{v}_s^2/c^2\right)^{\frac{1}{2}}. \tag{15}$$

[1] [19], p 300.

The arguments presented below are independent of the exact form of the velocity-squared $\hat{v}_s^2$ operator. However for completeness we may, following the classical relativistic relation between the velocity and momentum,

$$v_s^2/c^2 = p_s^2c^2/\left(p_s^2c^2 + E_s^2\right),$$

where $E_s = m_s c^2$ is the rest energy of body s, define such an operator. In order to guarantee its self-adjointness, we define a $\hat{v}_s^2$ operator within the usual rigged Hilbert space formulation of quantum mechanics [17, 18], as

$$\hat{v}_s^2/c^2 = \tfrac{1}{2}\left[\hat{p}_s^2c^2\left(\hat{p}_s^2c^2 + \hat{E}_s^2\right)^{-1} + \left(\hat{p}_s^2c^2 + \hat{E}_s^2\right)^{-1}\hat{p}_s^2c^2\right]. \tag{16}$$

For fixed t there are uncertainties associated with the proper times due to the quantum-mechanical uncertainties in $\hat{v}_s$. We also note that for $t > 0$, the spectrum of $\hat{t}_s$ is bounded from above by t (when $v_s = 0$) and from below by 0 (when $v_s = c$), with the upper bound becoming the lower bound and *vice versa* when $t < 0$. The 'proper-time' operator does not suffer from singularities when the momentum $p_s = 0$, besetting such operators as the non-relativistic 'measurement-time' or 'arrival-time' [22–25] or the 'tempus' operator [26].

It follows from equation (13) that $\dot{t}_s$, the classical *'velocity of time'* is given as

$$\dot{t}_s \equiv \mathrm{d}t_s/\mathrm{d}t = \left(1 - \beta_s^2\right)^{\frac{1}{2}}.$$

We note that for a constant velocity the transformed time of equation (14) yields the same form for the 'velocity of time'.

The Lagrangian of equation (2) of two non-interacting A and B particles can be written as

$$\mathcal{L} = -E_A\dot{t}_A - E_B\dot{t}_B,$$

where $E_{A(B)} = m_{A(B)}c^2$ is the rest energy of particle $A(B)$ in the reference frame. The classical (4th) momentum conjugate to t_s is obtained by the usual definition as

$$p_{t_s} = \partial\mathcal{L}/\partial\dot{t}_s = -E_s, \qquad s = A, B.$$

We can now canonically transform the proper times of bodies A and B to two new variables, defined as $t_{1(2)} = (t_A \pm t_B)/2$. Naturally $\dot{t}_{1(2)} = (\dot{t}_A \pm \dot{t}_B)/2$. Hence when we choose $E_A = E_B = E_s$

$$\mathcal{L} = -E_s(\dot{t}_A + \dot{t}_B) = -2E_s\dot{t}_1.$$

Therefore, the conjugate momenta to t_1 and t_2 are given as

$$p_{t_1} = \partial\mathcal{L}/\partial\dot{t}_1 = -2E_s p_{t_2} = \partial\mathcal{L}/\partial\dot{t}_2 = 0.$$

The invariant we are now seeking is simply $-2E_s$ since

$$-2E_s = p_{t_A} + p_{t_B} = p_{t_1} + p_{t_2}.$$

Because E_s is a function of p_{t_A} and p_{t_B} and also of p_{t_1} and p_{t_2}, $\hat{E}_s$, the (rest-energy) operator corresponding to it, commutes with all four momentum operators. Hence we can write the eigenstates of $-2\hat{E}_s$ in two different ways,

$$\langle t_A|p_{t_A}, n\rangle\langle t_B|p_{t_B}, n\rangle = \langle t_1|p_{t_1}, n\rangle\langle t_2|p_{t_2}, n\rangle, \tag{17}$$

where n is any other quantum number needed to specify the state. In the present construction we choose on purpose just a single n quantum number, which therefore remains the same upon execution of the canonical transformation. Henceforth, we omit the explicit mention of n but it should be considered as present.

Choosing two particular eigenvalues, $p_{t_A} = p_{t_B} \equiv -E_s$ we have that $p_{t_1} = -2E_s$, $p_{t_2} = 0$, with equation (17) now reading,

$$\langle t_A| - E_s\rangle\langle t_B| - E_s\rangle = \langle(t_A + t_B)/2| - 2E_s\rangle\langle t_A - t_B|0\rangle.$$

If we now also choose the particular values $t_A = t_B \equiv t_s$ we have that

$$\langle t_s| - E_s\rangle\langle t_s| - E_s\rangle = \langle t_s| - 2E_s\rangle\langle 0|0\rangle.$$

we obtain that

$$\langle t_s| - 2E_s\rangle' = (\langle t_s| - E_s\rangle')^2,$$

where

$$\langle t_s| - E_s\rangle' \equiv \langle t_s| - E_s\rangle/\langle 0|0\rangle.$$

It immediately follows, by the same type of arguments presented in section 1, that

$$\langle t_s| - E_s\rangle = a\exp(-E_s f(t_s)), \tag{18}$$

where $f(t_s)$ is only a function of t_s.

If instead of choosing $E_A = E_B = E_s$, we now choose $E_A = E_s$ and $E_B = 0$ (e.g., the second particle is a photon) we have that

$$\mathcal{L} = -E_s\dot{t}_A = -E_s(\dot{t}_1 + \dot{t}_2),$$

and that

$$p_{t_B} = 0 \qquad \text{and} \qquad p_{t_A} = p_{t_1} = p_{t_2} = -E_s.$$

It follows that

$$\langle t_A| - E_s\rangle\langle t_B|0\rangle = \langle t_1| - E_s\rangle\langle t_2| - E_s\rangle.$$

Choosing $t_A = 2t_s$ and $t_B = 0$ we have that $t_1 = t_s$ and $t_2 = t_s$, hence

$$\langle 2t_s| - E_s\rangle\langle 0|0\rangle = \langle t_s| - E_s\rangle\langle t_s| - E_s\rangle = (\langle t_s| - E_s\rangle)^2.$$

We now obtain that

$$\langle t_s| - E_s\rangle = a\exp(t_s g(-E_s)). \tag{19}$$

By equating equations (18) and (19) we obtain that $-g(-E_s)/E_s = f(t_s)/t_s = \alpha$, a constant, which means that $g(-E_s) = -\alpha E_s$ and

$$\langle t_s|E_s\rangle = a\exp(-\alpha E_s t_s). \tag{20}$$

We now show that the constant α must be a purely imaginary number. Because the $[0, t]$ boundaries of t_s are *finite*, we cannot argue, as we did in the coordinate–momentum case of section 1, where the boundaries were infinite, that α must be a purely imaginary number because otherwise the amplitude would diverge at the boundaries. Rather, in the present case, α must be purely imaginary to maintain Lorentz invariance. This is because if α had a real part the $|\langle t_s|E_s\rangle| = |a|\exp(-R_e(\alpha)E_s t_s)$ distribution would be maximal (minimal) at $t_s = t$, for $R_e(\alpha) > 0(R_e(\alpha) < 0)$, giving rise to different physical observations for different definitions of t, i.e., different reference frames, in clear contradiction to Lorentz invariance. Thus the constant α must be a purely imaginary number $\alpha = \mathrm{i}\gamma$. By again identifying γ with $1/\hbar$, (which is in fact simply the definition of the scaling of E_s) we obtain that,

$$\langle t_s|E_s\rangle = \langle 0|0\rangle\, \mathrm{e}^{-\mathrm{i}E_s t_s/\hbar}. \tag{21}$$

Three comments are now in order:

(1) The above derivation applies even to a truly structureless elementary particle for which the rest energy is just a single number. The reason is that in this case equation (19) still holds because t_s is definitely a continuous variable and the infinitesimal change in it, which is part of our proof, is perfectly permissible. Because there is now only a single value of E_s, the function $g(-E_s)$ now becomes a simple number g_s, and we define E_s as $E_s = -g_s\hbar$, with equation (21) immediately following.

(2) The above proof also holds for bound states. In this case, the energies may be varied by subjecting the particle to an external field and changing the strength of this external field. Once equation (21) is proved in the presence of the field, we can adiabatically switch off the field while establishing equation (21) for smaller and smaller external fields, until we reach the limit when the external field is zero.

(3) The spectrum of $\hat{t}_s$, depending on the spectrum of $\hat{v}_s$, extends from $t_s = 0$ to $t_s = t$. As pointed out by Galapon [16], for such a bounded time operator, in the rigged Hilbert space of scattering theory [17, 18] the objections raised by Pauli [15] against the existence of a self-adjoint time operator (having to do with the non-existence of a conjugate Hamiltonian operator with a point spectrum) do not apply.

We can now use equation (21), to construct the rest-energy operator in the proper-time representation as,

$$\begin{aligned}\langle t_s|\hat{E}_s|\Psi\rangle &= \int \mathrm{d}t' \langle t_s| \left\{ \sum_i E_i |E_i\rangle\langle E_i| + \int \mathrm{d}E\, E|E\rangle\langle E| \right\} |t'\rangle\langle t'|\Psi\rangle \\ &= |a|^2 \int \mathrm{d}t' \left\{ \sum_i E_i\, \mathrm{e}^{-\mathrm{i}E_i(t_s-t')/\hbar} + \int \mathrm{d}E\, E\, \mathrm{e}^{-\mathrm{i}E(t_s-t')/\hbar} \right\} \langle t'|\Psi\rangle \\ &= \mathrm{i}\hbar \frac{\partial}{\partial t_s} \int \mathrm{d}t' \langle t_s| \left\{ \sum_i |E_i\rangle\langle E_i| + \int \mathrm{d}E |E\rangle\langle E| \right\} |t'\rangle\langle t'|\Psi\rangle \\ &= \mathrm{i}\hbar \frac{\partial}{\partial t_s} \int \mathrm{d}t' \langle t_s|t'\rangle\langle t'|\Psi\rangle = \mathrm{i}\hbar \frac{\partial}{\partial t_s} \langle t_s|\Psi\rangle, \end{aligned} \tag{22}$$

where E_i are the discrete eigenvalues of $\hat{E}_s$. The $[\hat{t}_s, \hat{E}_s] = -\mathrm{i}\hbar$ commutation relation follows immediately from equation (22), leading in the usual fashion [3] to the corresponding uncertainty relations.

The $t_s = t$ eigenvalue is of special interest because it occurs when $v_s = 0$, i.e., when we equate the s-inertial frame with the reference frame. In this case $\hat{E}$ the total-energy operator and $\hat{E}_s$ the rest-energy operator coincide. We thus have as a special case of equation (21) that

$$\langle t|E\rangle = a\, \mathrm{e}^{-\mathrm{i}Et/\hbar}. \tag{23}$$

and as a special case of equation (22) that

$$\langle t|\hat{E}|\Psi\rangle = \mathrm{i}\hbar \frac{\partial}{\partial t} \langle t|\Psi\rangle. \tag{24}$$

Note however that t in contrast to $\hat{t}_s$ is a number and not an operator. We have thus obtained the quantum-mechanical evolution equations (equation (1)).

In special relativity, the substitution of the $\hat{E}^2 = \hat{E}_s^2 + \hat{p}_s^2 c^2$ on the rhs of equation (1), together with equation (12), leads to the time-dependent Klein–Gordon [27] or Dirac [20, 27] equations. In the non-relativistic limit the substitution of $\hat{E} = \hat{p}^2/2m$ together with equation (12) leads to the time-dependent Schrödinger equation.

4. Extensions to accelerating systems

So far we have treated inertial systems in which v_s was constant. For accelerating systems for which $v_s(t)$ is a non-constant function of time, the definition of the proper time must be modified. Realizing that the speed of light in any frame must still be conserved, the differential relation

$$\mathrm{d}t_s = \mathrm{d}t\left(1 - v_s^2(t)/c^2\right)^{\frac{1}{2}} \tag{25}$$

still holds. This means that

$$t_s = \int_0^t \mathrm{d}t' \big(1 - v_s^2(t')/c^2\big)^{\frac{1}{2}}. \quad (26)$$

Alternatively we can consider the differential transformed time

$$\mathrm{d}t_s = (\mathrm{d}t - \mathrm{d}x_s\, v_s(t)/c^2)\big/\big(1 - v_s^2(t)/c^2\big)^{\frac{1}{2}}. \quad (27)$$

In either case, the *velocity of time* $\dot{t}_s$ remains the same because in either differential equation for $\mathrm{d}t_s$, we have that

$$\dot{t}_s = \big(1 - v_s^2(t)/c^2\big)^{\frac{1}{2}}. \quad (28)$$

Thus our entire analysis expressed in proper time and rest energy is correct for non-inertial systems as is and we obtain that

$$\langle t_s|\hat{E}_s|\Psi\rangle = \mathrm{i}\hbar \frac{\partial}{\partial t_s}\langle t_s|\Psi\rangle. \quad (29)$$

However we can no longer make the transition to the total energy and the 'time' because the system linked to the particle is no longer an inertial system and cannot serve as a 'reference frame'.

We note that for accelerating frames, $\langle t_s|E_s\rangle$, although remaining the same as a function of t_s, assumes a different form as a function of t. We have that

$$\begin{aligned}\langle t_s|E_s\rangle &= a\exp(-\mathrm{i}E_s t_s/\hbar) = a\exp\left(-\mathrm{i}E_s\int_0^t \mathrm{d}t'\big(1 - v_s^2(t')/c^2\big)^{\frac{1}{2}}/\hbar\right)\\ &= a\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\,\mathcal{L}(t')/\hbar\right).\end{aligned} \quad (30)$$

The motion no longer represents a plane wave in spacetime but a motion in a curved space! It is instructive to derive a semi-classical relativistic equation for the motion of an accelerating particle. Using the relation $E(t') = E_s\big[1 - v_s^2(t')/c^2\big]^{-\frac{1}{2}}$ we have that

$$\begin{aligned}\exp(-\mathrm{i}E_s t_s/\hbar) &= \exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' E(t')\big[1 - v_s^2(t')/c^2\big]/\hbar\right)\\ &= \exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' E(t')/\hbar\right)\exp\left(\mathrm{i}\int_0^t \mathrm{d}t' E(t') v_s^2(t')\big/(c^2\hbar)\right).\end{aligned} \quad (31)$$

Dropping the s index as applying to the velocity and momentum, and using the definition $v(t') = \mathrm{d}x'/\mathrm{d}t'$ and the classical relativistic relation $p(x') = E(t')v(t')/c^2$, we have that

$$\exp(-\mathrm{i}E_s t_s/\hbar) = \exp\left(-\mathrm{i}\int_0^t \mathrm{d}t' E(t')/\hbar\right)\exp\left(\mathrm{i}\int_0^x \mathrm{d}x' p(x')/\hbar\right). \quad (32)$$

We have thus derived a joint spatio-temporal semiclassical form of the relativistic wavefunction in curved spaces.

5. Discussion

In this paper we have derived the special-relativistic energy–time transformation matrices and the quantum evolution equations. A dynamical quantum-mechanical equation valid for accelerating frames was also derived. The derivation involved the *proper-time* operator, defined as a function of $\hat{v}_s$, the velocity operator, and depending parametrically on t, the time measured in a selected inertial frame, the so-called reference frame.

The present derivation is the only one to our knowledge where the quantum-dynamical equations are derived from *within* conventional quantum mechanics, making no new postulates. It relies only on the basic structure of the rigged Hilbert space of quantum mechanics; on the linear correspondence between observables and quantum operators; on canonical invariance with respect to transformation between pairs of conjugate variables; and on Lorentz invariance. It is interesting to note that the only aspect of canonical invariance used here is that the canonically transformed conjugate pair of variables describe the same point in phase space as the untransformed conjugate pair.

Applications of equations (29)–(32) to quantum dynamics of highly accelerating systems subject to gravitational potentials are considered elsewhere [28].

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