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Twistor Primer

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Abstract. An abstract should be given

1. Introduction

In the past 30 years, a lot of work has been done on developing twistor theory. Its creator, Roger Penrose, was first led to the concept of twistors in his investigation of the structure of spacetime and it was he who first saw the wide range of applications for this new mathematical construct. Yet 30 years later, twistors remain relatively unknown even in the mathematical physics community. The reason for this may be the air of mystery that seems to surround the subject even though it provides a very elegant formalism for both general relativity and quantum theory. These notes are based on a graduate lecture course given by R. Penrose in Mathematical Institute, Oxford, in 1997 and should give a brief introduction to the basic definitions. Let us begin with the building blocks: spinors.

Spinors

2. Twistors

Let be a vector in a spacetime. We choose to represent it in the form of a Hermitian matrix:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} K^0+K^3 & K^1+i\,K^2 \\ K^1-i\,K^2 & K^0-K^3 \end{pmatrix}$$

Note that . If is null future-pointing, there is a decomposition

, ,

i.e. . Now is a spinor and is an element of the conjugate spin space.

The notation may be unusual, but all we are doing is elementary algebra. We have shown that we can parameterize the null cone by 2 complex numbers, the components of the spinor . In the projective case, when we are interested only in the direction of , the space of parameterization reduces to , the 1-dimensional complex projective space, which is homeomorphic to , the celestial sphere.

Since det is (up to a constant factor) the unique skew-symmetric form of maximal rank, we have the following decomposition:

where , with all diagonal entries zero.

Now we can define raising and lowering of spinor indices (nb., order of indices is important since fs are skew-symmetric!):

Spinor calculations are equivalent to tensor calculations in many ways. The main difference is that in the spinor case there are unprimed and primed (conjugate) spaces with their respective duals. The isomorphisms between the spaces and their duals are given by the skew-symmetric form so care must be taken to write the indices in the correct order.

Note that since fs are skew.

3. Electromagnetism in spinor notation

We present here the basic results written in spinor notation. You can prove all of them as an exercise.

Every skew can be written as , where and are symmetric. If is real then . Dualisation is particularly easy:

Maxwellfs equations have the form:

for some 4-potential and 4-current . The expression for the energy-momentum tensor is also particularly simple:

4. Twistors

One of the easiest and most straightforward ways of defining twisters uses the transformation properties of linear and angular momentum of a particle under a shift of origin. Consider a change of origin from 0 to a point Q with coordinates . With respect to the new origin,

We define the Pauli-Lubanski spin vector:

It is easy to show that . Now let be future null, so that we can write . Since is skew it can be decomposed as

The dual is then easy to write:

and

In nature we only observe massless particles with definite handedness, i.e. with

follows immediately and hence , where either or are proportional to (note that always) and () denotes index symmetrization.

The same argument applies to . We can now define with

At last, we can say,

is a twistor.

Now we have

if we define .

5. Quantization

Canonical commutation rules for Minkowski spacetime

induce the following commutation relations on the twistor space:

One way of quantizing the theory is to use the following substitution:

The spin operator can be easily derived in the non-commutative case following the same procedure. The result is the symmetrized form:

.

Therefore, if we want a twistor function to be an eigenstate of spin operator with eigenvalue must be homogeneous of degree -2s-2.

6. Klein Correspondence

In this section we outline some basic twistor geometry. Let be complexified compactified Minkowski space. (We can think of it as the Grassmanian manifold .) The basic concept is that of incidence.

Twistor is incident with a spacetime point iff

.

Dual twistor is incident with a spacetime point iff

. is incident with iff .

In the projective twistor space defines a point as an equivalence class of all twistors in T proportional to . The set of all spacetime points incident with it forms a totally null complex 2-plane in M (a plane). If we denote the incidence relation with , then

,

for all . Any two vectors in the a-plane are orthogonal to each other. Since and have the same incidence properties for , it is natural to study incidence on the projective twistor space .

Dual twistors now correspond to planes in PT and are incident on b-planes in M. These are also totally null complex 2-planes in M.

Finally, it can be shown that iff there is a spacetime point incident on both of them. Geometric correspondence defined by the incidence relation (Klein correspondence) is summarized in the following table:

PT M

point line point -plane

line plane point -plane

point plane complex light ray

7. Classical Fields

Let be a spin n/2 field and

be a spin -n/2 field on M, where n is the number of spinor indices on s.

Zero mass field equations are given by

or for zero spin case.

Solutions then can be written in the form:

As an exercise, show that for

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